A Modified Kolmogorov-Smirnov Test for Normality

Zvi Drezner, Ofir Turel and Dawit Zerom*
Steven G. Mihaylo College of Business and Economics
California State University-Fullerton
Fullerton, CA 92834.

Abstract

In this paper we propose an improvement of the Kolmogorov-Smirnov test for normality. In the current implementation of the Kolmogorov-Smirnov test, given data are compared with a normal distribution that uses the sample mean and the sample variance. We propose to select the mean and variance of the normal distribution that provide the closest fit to the data. This is like shifting and stretching the reference normal distribution so that it fits the data in the best possible way. A study of the power of the proposed test indicates that the test is able to discriminate between the normal distribution and distributions such as uniform, bi-modal, beta, exponential and log-normal that are different in shape, but has a relatively lower power against the student t-distribution that is similar in shape to the normal distribution. We also compare the performance (both in power and sensitivity to outlying observations) of the proposed test with existing normality tests such as Anderson-Darling and Shapiro-Francia.

Keywords: Closest fit; Kolmogorov-Smirnov; Normal distribution.

1 Introduction

Many data analysis methods depend on the assumption that data were sampled from a normal distribution or at least from a distribution which is sufficiently close to a normal distribution. For example, one often tests normality of residuals after fitting a linear model to the data in order to ensure the normality assumption of the model is satisfied. Such an assumption is of great importance because, in many cases, it determines the method that ought to be used to estimate the unknown parameters of a model and also dictates the test procedures which analysts may apply. There are several tests available to determine if a sample comes from a normally distributed

*Corresponding author: Mihaylo College of Business and Economics, California State University, Fullerton, CA, 92834-6848, (657) 278-3635, dzerom@fullerton.edu.

population. Those theory-driven tests include the Kolmogorov-Smirnov test, Anderson-Darling test, Cramer-von Mises test, Shapiro-Wilk test and Shapiro-Francia test. The first three tests are based on the empirical cumulative distribution. Shapiro-Francia test (Shapiro and Francia, 1972 and Royston, 1983) is specifically designed for testing normality and is a modification of the more general Shapiro-Wilk test (Shapiro and Wilk 1965). There are also tests that exploit the shape of the distribution of the data. For example, the widely available Jarque-Bera test (Jarque and Bera, 1980) is based on skewness and kurtosis of the data. To complement the results of formal tests, graphical methods (such as box-plots and Q-Q plots) have also been used and increasingly so in recent years.

In this paper we focus on the Kolmogorov-Smirnov (KS) test. The KS test is arguably the most well-known test for normality. It is also available in most widely used statistical software packages. In its original form, the KS test is used to decide if a sample comes from a population with a completely specified continuous distribution. In practice, however, we often need to estimate one or more of the parameters of the hypothesized distribution (say, the normal distribution) from the sample, in which case the critical values of the KS test may no longer be valid. For the case of normality testing, Massey (1951) suggests using sample mean and sample variance, and this is the norm in the current use of KS test. Lilliefors (1967) and Dallal and Wilkinson (1986) provide a table of approximate critical values of the KS statistics which are based on sample mean and sample variance.

While the use of sample mean and sample variance seems a natural choice, using these values is not necessarily the best available option. When one concludes (after using the KS test) that given data are not normal, this only means that the data are not normal with the sample mean and sample variance. But it could well be that the data are normal or sufficiently close to normal at other values of the mean and variance of the normal distribution. Although the scope of this paper is limited to the KS test, this drawback is also shared by other tests such as Anderson-Darling and Cramer-von Mises tests. Interestingly, Stephens (1974) writes after comparing several tests (such as KS, Anderson-Darling and so on) "It appears that since one is trying, in effect, to fit a density of a certain shape to the data, the precise location and scale is relatively unimportant, and being tied down to fixed values, even correct ones, is more of a hinderance than a help." In this paper, we suggest an approach that circumvents the need to use sample mean and sample variance of given data. Instead, we look for mean and variance values such that the corresponding normal distribution provides the closest fit to the empirical distribution of the data. When the KS test with such values still shows deviation from normality, we conclude that the data are probably not sampled from a normal distribution.

Suppose that the sample consists of n independent observations. These observations are sorted $x_1 \leq x_2 \leq \ldots \leq x_n$. The cumulative distribution of the data is a step function where the step is between $\frac{k-1}{n}$ and $\frac{k}{n}$ at each x_k. On the other hand, for given mean μ and variance σ^2, the cumulative normal distribution at x_k is $\Phi\left(\frac{x_k-\mu}{\sigma}\right)$. The KS statistics is defined as

$$KS(\mu,\sigma) = \max_{1\leq k\leq n} \left\{ \frac{k}{n} - \Phi\left(\frac{x_k-\mu}{\sigma}\right), \Phi\left(\frac{x_k-\mu}{\sigma}\right) - \frac{k-1}{n} \right\}. \tag{1}$$

The traditional KS statistics is simply $KS(\bar{x}, s)$. We propose a modified KS statistics denoted by $KS(\tilde{\mu}, \tilde{\sigma})$ where the vector $(\tilde{\mu}, \tilde{\sigma})$ is a solution to the following minimization problem

$$\min_{\mu,\sigma} \{KS(\mu,\sigma)\} \tag{2}$$

where $KS(\mu,\sigma)$ is as given in (1). In the Appendix, we analyze this optimization problem and provide a tractable algorithm for its solution. The proposed algorithm is quite efficient and we are able to compute the critical values of the modified KS test using 100 million replications (see Table 1) in less than 4 days, i.e. 6000 calculations per second.

To the best of our knowledge there has not been any study that extends the KS test by allowing the use of optimized distribution parameters. Closely related to our approach is the work of Weber *et al* (2006) in which they consider the problem of parameter estimation of continuous distributions (not just normal distribution) via the minimization of the KS statistics. They use the heuristic optimization algorithm of Sobieszczanski-Sobieski *et al* (1998) to estimate the parameters of a number of widely used distributions and also provide a user-friendly software tool. The practical advantage of this software is that it suggests a best fitted distribution to given data by looking at the minimized KS statistics values among a set of continuous distributions. In this sense, our algorithm of minimizing the KS statistics may also serve the same purpose as that of Weber *et al* (2006) although our paper is wider in scope.

In Section 2 we compute critical values for the modified KS statistics using Monte Carlo simulation. Section 3 provides comparison of the approximate powers of the modified KS test with the traditional KS test (and other existing tests) for a set of selected distributions. We also investigate the sensitivity of the various tests to the presence of moderately outlying observation. Section 4 concludes. In the Appendix we provide a tractable algorithm for implementing the modified KS statistics.

2 Monte Carlo estimation of test statistics distribution

Monte Carlo simulation is used to compute critical values for the modified KS statistics. We draw a random sample of size n from a standard normal distribution and estimate $\tilde{\mu}$, $\tilde{\sigma}$ and $KS(\tilde{\mu}, \tilde{\sigma})$. For every n, we repeat this procedure 100 million times. The critical values are given in Table 1. We also recalculate the critical values for the traditional KS test in the same way and they are available in Table 1. Because we use 100 million samples, the critical values we report for the traditional KS test are more accurate than Lilliefors (1967) and Dallal and Washington (1986).

`Table (1) about here`

The critical values for both KS tests can also be approximated for $n \geq 20$ by the formula $a + \frac{b}{\sqrt{n}}\left(1 - \frac{c}{n}\right)$ where a, b and c are functions of α. These three parameters are given in Table 2. The approximation is very accurate with an error (when compared to Table 1) of not more than 0.0002. Thus, the approximation formula can replace the tables for $n \geq 20$. We obtain the approximation formula via multiple regression, where for each α, the critical values in Table 1 are used as the dependent variable, and $\frac{1}{\sqrt{n}}$ and $\frac{1}{n\sqrt{n}}$ are the independent variables. We select these two independent variables through experimentation. We begin with a single variable regression involving only $\frac{1}{\sqrt{n}}$. We then add variables, one at a time, which are functions of n. A regression involving $\frac{1}{\sqrt{n}}$ and $\frac{1}{n\sqrt{n}}$ provides an excellent fit. For example, the adjusted R-squared is greater than 99% for all α values reported in Table 2.

`Table (2) about here`

3 Numerical results

In this section we use Monte Carlo simulation to evaluate the relative performance of the proposed test among existing tests of normality. First, we compare the approximate power of the modified KS test with the traditional KS test for a set of selected distributions that convey a wide array of shapes where some resemble the normal distribution while others are substantially different. To put the power comparison between the modified KS and traditional KS tests within the context of other existing tests of normality, we also include the following four tests that are widely available in statistical packages: Anderson-Darling (AD), Cramer-von Mises (CVM), Shapiro-Francia (SF) and Pearson chi-square (PEARSON). Second, we explore how the above tests of normality behave (in terms of their rejection probabilities or size) in the presence of moderately outlying observations. In practice, researchers often deal with small data sets with potentially few outlying (moderate or extreme) observations while much of these data may well be approximately normal. In situations like this, a particular test may suggest rejection of normality implying the potential need for transformations or complex models.

3.1 Power comparisons

In the power comparisons we consider a uniform (0,1) distribution; a bi-modal distribution which is a composite of two normal distributions, one centered at +2 and one at -2 with variance of 1; a beta(1,2) distribution whose density function is a straight line connecting (0, 0) and (1, 1); an exponential distribution with mean and variance of 1; a log-normal distribution with mean $e^{1/2}$ and variance $e(e-1)$ and three t-distributions with degrees of freedom 1, 2 and 6. Some of these distributions are also used in Lilliefors (1967) and Stephens (1974), among others. For a given alternative hypothesis (say, a uniform distribution), computation of the power of the modified KS test is done as follows. We draw a random sample of size n ($n = 10, 20, \cdots, 100$) from the distribution specified in the alternative hypothesis. Based on this sample, we estimate the parameters $\tilde{\mu}$ and $\tilde{\sigma}$ using the algorithm outlined in Appendix and compute $KS(\tilde{\mu}, \tilde{\sigma})$. Then, we apply the critical values in Table 2 to test if such sample comes from a normal distribution. Repeating this procedure 10,000 times, and counting the number of correct decisions gives the approximate power. The same approach is followed for computing power for traditional KS, AD, CVM, SF and PEARSON tests. We use the library NORTEST available in the software R to implement AD, CVM, SF and PEARSON. For better exposition we present the complete power results in graphic form in Figure 1. To save space, we only report results for $\alpha = 0.05$ (the behavior is very similar for other values of α).

From Figure 1 we can see that the power of the modified KS test is consistently better than both the traditional KS and PEARSON tests for uniform, beta and bi-modal distributions. For beta and bi-modal shapes, the modified KS also performs reasonably well overall in comparison to AD, SF and CVM tests, especially for $n \leq 30$. For exponential and log-normal distributions, the power of the modified KS test is lower than the traditional KS test for $n \leq 30$ although the two powers converge when $n \geq 40$. For these two distributions, all the considered tests have similar powers for $n \geq 50$. At smaller samples, the powers of the two KS tests lag behind other tests. For the t-distributions, the modified KS test has the lowest power among all the tests including the traditional KS test. For t_1 case, we excluded CVM from the analysis because it is very sensitive to large outliers and increasingly so for large sample sizes. This effect is also slightly reflected in t_2 where the power start to decrease for $n > 80$. What is common to the t-distributions is that they resemble the normal distribution except for their heavier tails. In theory, with increasing degrees of freedom, the tails of the t-distribution get lighter and eventually behave like a normal distribution. The modified KS test has difficulty detecting non-normality when the observed distribution is similar to normal and increasingly so with larger degrees of freedom, i.e. as it gets closer to normal.

`Figure (1) about here`

On the surface, the low power for the t-distribution may seem like a weakness of the modified KS test. However, would one expect, with a small n, that data generated by a t_6 distribution be distinguishable from a normal distribution - thus be identified as non-normal? We argue that the reason the traditional KS test and the other tests have better power is that they rejects data which can be fitted quite well to a normal distribution by a proper selection of μ and σ. It is surprising that the power of the traditional KS test is higher for a t_2 distribution than it is for the uniform and beta distributions while the latter are substantially different from normality. In contrast, the modified KS test tries to look for those mean and variance values that lead to the closest fit to the data. In a way, we are trying to approximate the reference distribution (the t-distribution) with a normal distribution. If such a normal approximation exists, the data may be considered sufficiently normal. For example, for t_6, the powers of the modified KS are close to $\alpha = 0.05$ implying the sample data is hardly distinguishable from the normal distribution. When the degrees of freedom is made smaller, the power of the modified KS test improves because the deviation from normality gets larger. When normal approximation can not be achieved, the sample data is flagged as non-normal. For t_2, the modified KS test requires about $n = 200$ (not shown in the graphs) to detect non-normality while t_6 requires an extremely large n to be detected by the modified KS test. For t_1, the power of the proposed KS test gets a lot better reaching decent power at $n = 100$. The reason is that t_1 has a much heavier tail than the normal distribution making normal approximation via adapting mean and variance values very difficult.

To summarize, the modified KS test is able to better discriminate between the normal distribution and those distributions that are very different in shape from normal. For such distributions that substantially deviate from normality, the modified KS test has an overall better power than the traditional KS test while performing reasonably well among existing tests. However, the modified KS test has the weakest power among all tests in detecting non-normality when the shape of a distribution resembles a normal distribution.

3.2 Sensitivity of tests

Here we evaluate all the tests included in the above power comparison in terms of their size sensitivity (at $\alpha = 0.05$). We consider $n = 10, 20, \ldots, 100$ (in intervals of 10). For each n, we generate 10,000 standard normal random observations. If a test is correctly sized, we expect this percentage to be close to 5%. Size is defined as the percentage of times (out of the total 10,000 samples) a test rejects the null hypothesis of normality. We also analyze (for each n) two adjustments of the normally generated data where we randomly replace one of the n observations by a constant C: (1) C=3 and (2) C=3.25. Although these two values are considered outlying observations for a standard normal case, they are not necessarily uncommon and the purpose is to mimic what can happen in practice. Especially for n that is relatively large, we wish the size of a test to be insensitive to the presence of outlying observations.

Using 10,000 replications, we plot the size of all the tests in Figure 2. When no outlying observation is added, all tests have sizes close to 5% although PEARSON and SF seem to over-reject relative to the other tests. Interestingly, when an outlying observation is included, the modified KS test is quite insensitive and remains close to the 5% level while all other tests over-reject even for a relatively large n. With the observed pattern, it will take a very large n to alleviate the effect of the outlying observations which are not even that extreme. In practice, researchers often deal with small data sets that include one or more observations that contribute to the rejection of normality when existing tests of normality are applied. The above analysis shows that existing tests of normality are sensitive to even moderately outlying observations. In contrast, the modified KS test is fairly robust and can lead to a more nuanced judgments regarding normality.

`Figure (2) about here`

4 Conclusion

Many data analysis methods (t-test, ANOVA, regression) depend on the assumption that data were sampled from a normal distribution. One of the most frequently used tests to evaluate how far data are from normality is the Kolmogorov-Smirnov (KS) test. In implementing the KS test, statistical software packages use the sample mean and sample variance as the parameters of the normal distribution. We propose a modified KS test in which we optimally choose the mean and variance of the normal distribution by minimizing the KS statistics. Power comparison with the traditional KS test show that the modified KS test is better than the traditional KS test for distributions that are substantially different in shape from the normal distribution, and performs reasonably well among other existing normality tests (i.e. Anderson-Darling, Cramer-von Mises, Shapiro-Francia, and Pearson chi-square). However, the modified KS test is rather weak in detecting non-normality for those distributions that resemble normality. We also explore the sensitivity of the various tests of normality (in terms of their rejection probabilities or size) to the presence of moderately outlying observations. The modified KS test is fairly robust while all other tests are very sensitive, leading to over-rejection even for large samples.

One possible direction for future research is to extend our idea to the Anderson-Darling (AD) test. Because the current implementation of the AD test uses the sample mean and sample variance as parameters of the normal distribution, it shares the same shortcoming of the traditional KS test. For example, one can modify the AD test by using mean and variance values that minimize the AD statistics and create a corresponding critical values table. Because the AD test is among the most powerful available tests, such simple modification can make it even more attractive for practitioners by further increasing its power and making it less sensitive to outlying observations.

Acknowledgments

We thank the associate editor and two anonymous referees, whose comments and suggestions greatly improved the presentation of the paper.

References

1. Dallal G. E. and L. Wilkinson (1986) "An analytic approximation to the distribution of Lilliefors's test statistic for normality," *The American Statistician*, 40, 294-296.

2. Jarque, C.M. and A.K. Bera (1980) "Efficient Tests for Normality, Homoscedasticity and Serial Independence of Regression Residuals," *Economics Letters*, 6(3), 255-259.

3. Lilliefors H. W. (1967) "On the Kolmogorov-Smirnov test for normality with mean and variance unknown," *Journal of the American Statistical Association*, 62, 399-402.

4. Massey F. J. (1951) "The Kolmogorov-Smirnov test for goodness of fit," *Journal of the American Statistical Association*, 46, 68-78.

5. Megiddo N. (1983) "Linear-time algorithms for linear programming inR^3 and related problems, *SIAM Journal on Computing*, 12, 759-776.

6. Royston, J. P. (1983) "A Simple Method for Evaluating the Shapiro-Francia W' Test of Non-Normality," *Statistician*, 32(3) (September), 297-300.

7. Shapiro, S. S. and R. S. Francia (1972) "An Approximate Analysis of Variance Test for Normality," *Journal of the American Statistical Association*, 67, 215-216.

8. Shapiro, S. S. and M. B. Wilk (1965) "An Analysis of Variance Test for Normality (Complete Samples)," *Biometrika*, 52(3/4) (December), 591-611.

9. Sobieszczanski-Sobieski, J., Laba, K. and R. Kincaid (1998) "*Bell-curve evolutionary optimization algorithm*," Proceedings of the 7th AIAA Symposium on Multidisciplinary Analysis and Optimization, St. Louis, MO, 2-4 September, AIAA paper 98-4971.

10. Stephens, M.A. (1974) "EDF statistics for goodness of fit and some comparisons, " *Journal of the American Statistical Association*, 69, 730-737.

11. Weber, M., Leemis, L. and R. Kincaid (2006) "Minimum Kolmogorov-Smirnov test statistic parameter estimates," *Journal of Statistical Computation and Simulation*, 76, 3, 195-206.

Appendix: Algorithm

Here, we analyze the optimization problem given in equation (2) and provide a tractable algorithm for its solution. By (1)

$$KS(\mu,\sigma) \geq \frac{k}{n} - \Phi\left(\frac{x_k-\mu}{\sigma}\right)$$

$$KS(\mu,\sigma) \geq \Phi\left(\frac{x_k-\mu}{\sigma}\right) - \frac{k-1}{n}$$

Let L be the minimum possible value of $KS(\mu,\sigma)$. The solution to the following optimization problem is the minimum possible $KS(\mu,\sigma)$ and thus is equivalent to (2).

$$\min\{\ L\ \} \tag{3}$$

subject to:

$$\frac{k}{n} - \Phi\left(\frac{x_k-\mu}{\sigma}\right) \leq L \quad \text{for } k > nL \tag{4}$$

$$\Phi\left(\frac{x_k-\mu}{\sigma}\right) - \frac{k-1}{n} \leq L \quad \text{for } k < n(1-L)+1. \tag{5}$$

Note that if $\frac{k}{n} - L \leq 0$, constraint (4) is always true and if $L + \frac{k-1}{n} \geq 1$, constraint (5) is always true. We can solve (3-5) by designing an algorithm that finds whether there is a feasible solution to (4-5) for a given L.

For a given L, the constraints are equivalent to:

$$\mu \leq x_k - \Phi^{-1}\left(\frac{k}{n} - L\right)\sigma \quad \text{for } k > nL \tag{6}$$

$$\mu \geq x_k - \Phi^{-1}\left(L + \frac{k-1}{n}\right)\sigma \quad \text{for } k < n(1-L)+1. \tag{7}$$

Constraints (6) and (7) can be combined into one constraint each.

$$\mu \leq \min_{k>nL}\left\{x_k - \Phi^{-1}\left(\frac{k}{n} - L\right)\sigma\right\} \tag{8}$$

$$\mu \geq \max_{k<n(1-L)+1}\left\{x_k - \Phi^{-1}\left(L + \frac{k-1}{n}\right)\sigma\right\} \tag{9}$$

For a given σ there is a solution for μ satisfying the system of equations (8-9) if and only if

$$\min_{k>nL}\left\{x_k - \Phi^{-1}\left(\frac{k}{n} - L\right)\sigma\right\} \geq \max_{k<n(1-L)+1}\left\{x_k - \Phi^{-1}\left(L + \frac{k-1}{n}\right)\sigma\right\} \tag{10}$$

or

$$F(\sigma, L) = \min_{k>nL}\left\{x_k - \Phi^{-1}\left(\frac{k}{n} - L\right)\sigma\right\} - \max_{k<n(1-L)+1}\left\{x_k - \Phi^{-1}\left(L + \frac{k-1}{n}\right)\sigma\right\} \geq 0. \tag{11}$$

For a given L, the function $F(\sigma, L)$ is a piece-wise linear concave function in σ (see Figure 3). We prove that $F(\sigma, L)$ is a concave function in σ for a given L.

Theorem 1: *The function $F(\sigma, L)$ for a given L is concave in σ.*

Proof: All the functions in the braces of (11) are linear in σ and all the other values are constants for a given L. Furthermore, the minimum of linear functions is concave and the maximum of linear functions is convex. Therefore, the difference $F(\sigma, L)$ is a concave function in σ. □

By Theorem 1, for a given L, $F(\sigma, L)$ has only one local maximum which is the global one. The maximum value of $F(\sigma, L)$ for a given L can be easily found by a search on σ. For any value of σ $F(\sigma, L)$ can be calculated and if the slope is positive we know that the optimal σ is to the right, and if it is negative we know that it is to the left. The solution is always at the intersection point between two lines, one with a positive slope and one with a negative slope (see figure 3). Megiddo (1983) suggested a very efficient method for solving such a problem.

`Figure (3) about here`

Note that if $F(\sigma, L) \geq 0$, any μ in the range

$$\left[\max_{k<n(1-L)+1}\left\{x_k - \Phi^{-1}\left(L + \frac{k-1}{n}\right)\sigma\right\}, \min_{k>nL}\left\{x_k - \Phi^{-1}\left(\frac{k}{n} - L\right)\sigma\right\}\right]$$

(or specifically the midpoint of the range) with the σ used in calculating $F(\sigma, L)$ yields a KS statistic which does not exceed L.

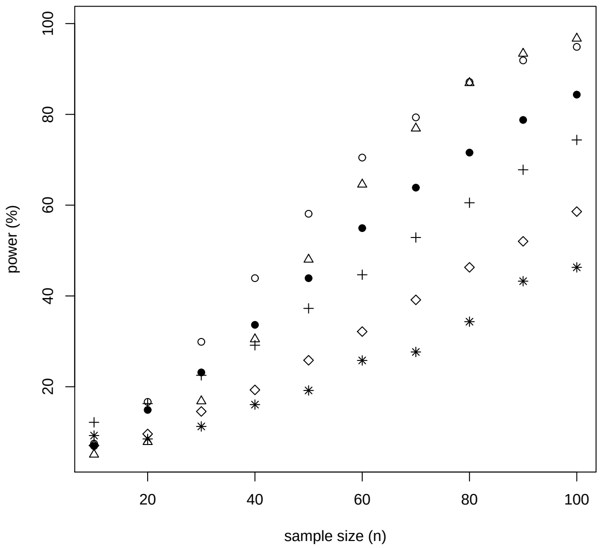
Let $G(L) = \max_{\sigma}\{F(\sigma, L)\}$ found by either the method in Megiddo (1983) or any other search method. If $G(L) \geq 0$, there is a solution (μ, σ) for this value of L and if $G(L) < 0$ no such solution exists. To find the minimum value of L we propose a binary search. The details of the binary search are now described. The optimal L must satisfy $L \leq KS(\bar{x}, s)$. Also, any KS statistic must be at least $\frac{1}{2n}$. Therefore, $\frac{1}{2n} \leq L \leq KS(\bar{x}, s)$. A binary search on any segment $[a, b]$ is performed as follows. $G(L)$ for $L = \frac{a+b}{2}$ is evaluated. If $G(L) \geq 0$, there is a solution (μ, σ) for this value of L and the search segment is reduced to $[a, \frac{a+b}{2}]$. If $G(L) < 0$ no such solution exists and the search segment is reduced to $[\frac{a+b}{2}, b]$. In either case the search segment is cut in half. Following a relatively small number of iterations, the search segment is reduced to a small enough range (such as 10^{-5}) and the upper limit of the range yields a solution (μ, σ) and its value of L is within a given tolerance (the size of the final segment) of the optimal value of L.

Table 1: Critical Values for the Traditional and Modified KS Test

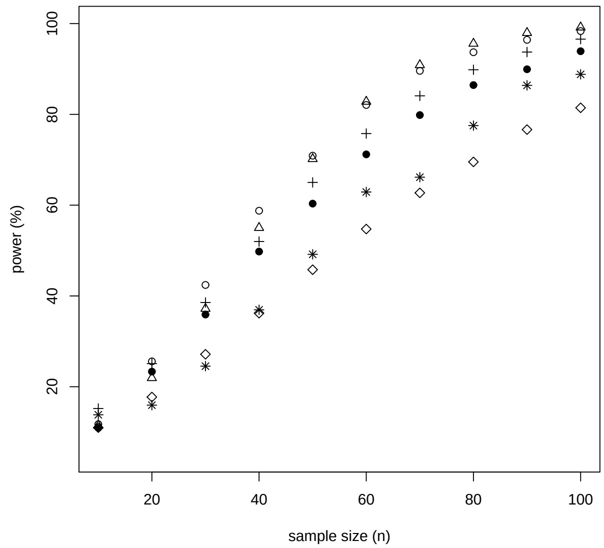
n	Traditional KS statistics						Modified KS statistics					
	Upper Tail Probabilities						Upper Tail Probabilities					
	0.20	0.15	0.10	0.05	0.01	0.001	0.20	0.15	0.10	0.05	0.01	0.001
4	0.3029	0.3215	0.3453	0.3753	0.4131	0.4327	0.2396	0.2436	0.2474	0.2499	0.2987	0.3518
5	0.2894	0.3027	0.3189	0.3430	0.3967	0.4388	0.2000	0.2108	0.2255	0.2458	0.2763	0.3063
6	0.2687	0.2809	0.2971	0.3234	0.3705	0.4232	0.1962	0.2046	0.2147	0.2286	0.2570	0.2945
7	0.2523	0.2643	0.2802	0.3042	0.3508	0.4011	0.1855	0.1922	0.2006	0.2139	0.2435	0.2708
8	0.2388	0.2503	0.2651	0.2880	0.3328	0.3827	0.1748	0.1810	0.1899	0.2038	0.2281	0.2502
9	0.2272	0.2381	0.2522	0.2741	0.3172	0.3657	0.1661	0.1727	0.1811	0.1932	0.2151	0.2418
10	0.2171	0.2274	0.2410	0.2621	0.3035	0.3509	0.1591	0.1650	0.1725	0.1836	0.2045	0.2324
11	0.2081	0.2181	0.2312	0.2514	0.2914	0.3375	0.1524	0.1578	0.1648	0.1753	0.1972	0.2240
12	0.2003	0.2099	0.2224	0.2420	0.2807	0.3255	0.1462	0.1514	0.1580	0.1681	0.1902	0.2158
13	0.1932	0.2025	0.2146	0.2335	0.2710	0.3146	0.1407	0.1457	0.1521	0.1627	0.1839	0.2081
14	0.1869	0.1958	0.2076	0.2259	0.2623	0.3048	0.1358	0.1406	0.1472	0.1576	0.1780	0.2012
15	0.1811	0.1898	0.2012	0.2189	0.2543	0.2958	0.1314	0.1363	0.1428	0.1528	0.1725	0.1949
16	0.1759	0.1843	0.1954	0.2126	0.2471	0.2875	0.1276	0.1325	0.1388	0.1485	0.1674	0.1893
17	0.1710	0.1793	0.1900	0.2068	0.2404	0.2800	0.1243	0.1290	0.1351	0.1445	0.1628	0.1845
18	0.1666	0.1746	0.1851	0.2015	0.2342	0.2729	0.1211	0.1257	0.1316	0.1407	0.1585	0.1799
19	0.1625	0.1703	0.1806	0.1965	0.2285	0.2663	0.1182	0.1226	0.1284	0.1372	0.1545	0.1756
20	0.1587	0.1663	0.1763	0.1919	0.2232	0.2603	0.1154	0.1198	0.1254	0.1339	0.1510	0.1716
25	0.1430	0.1499	0.1589	0.1730	0.2014	0.2351	0.1040	0.1079	0.1129	0.1207	0.1363	0.1547
30	0.1312	0.1376	0.1458	0.1588	0.1849	0.2161	0.0955	0.0990	0.1036	0.1108	0.1251	0.1422
40	0.1145	0.1200	0.1272	0.1385	0.1614	0.1889	0.0833	0.0864	0.0905	0.0967	0.1092	0.1242
50	0.1029	0.1078	0.1143	0.1245	0.1450	0.1699	0.0749	0.0777	0.0813	0.0869	0.0982	0.1116
60	0.0943	0.0988	0.1047	0.1140	0.1328	0.1556	0.0687	0.0712	0.0745	0.0797	0.0900	0.1023
70	0.0875	0.0917	0.0972	0.1058	0.1233	0.1445	0.0638	0.0661	0.0692	0.0740	0.0835	0.0950
80	0.0820	0.0859	0.0911	0.0992	0.1156	0.1355	0.0598	0.0620	0.0649	0.0694	0.0783	0.0891
90	0.0775	0.0812	0.0860	0.0937	0.1092	0.1279	0.0565	0.0586	0.0613	0.0655	0.0740	0.0841
100	0.0736	0.0771	0.0817	0.0890	0.1037	0.1216	0.0537	0.0557	0.0583	0.0623	0.0703	0.0799
400	0.0373	0.0390	0.0414	0.0450	0.0524	0.0615	0.0273	0.0283	0.0296	0.0316	0.0356	0.0405
900	0.0249	0.0261	0.0277	0.0301	0.0351	0.0411	0.0183	0.0190	0.0198	0.0212	0.0239	0.0271

Table 2: Coefficients for the approximate formulas

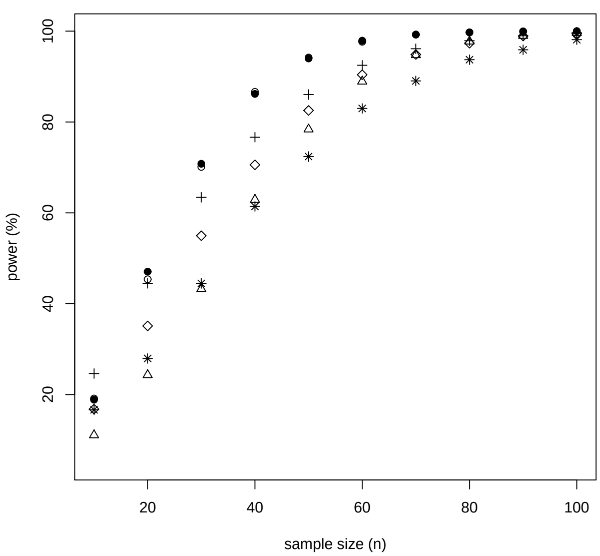
α	Traditional KS test			Modified Ks test		
	a	b	c	a	b	c
0.20	0.00053	0.73574	0.78520	0.00060	0.53446	0.80443
0.15	0.00049	0.77149	0.78515	0.00068	0.55329	0.76285
0.10	0.00059	0.81689	0.77062	0.00062	0.57999	0.78034
0.05	0.00052	0.89105	0.79780	0.00061	0.62082	0.81183
0.01	0.00054	1.03964	0.84912	0.00055	0.70276	0.85751
0.001	0.00052	1.22182	0.99171	0.00056	0.79997	0.89234



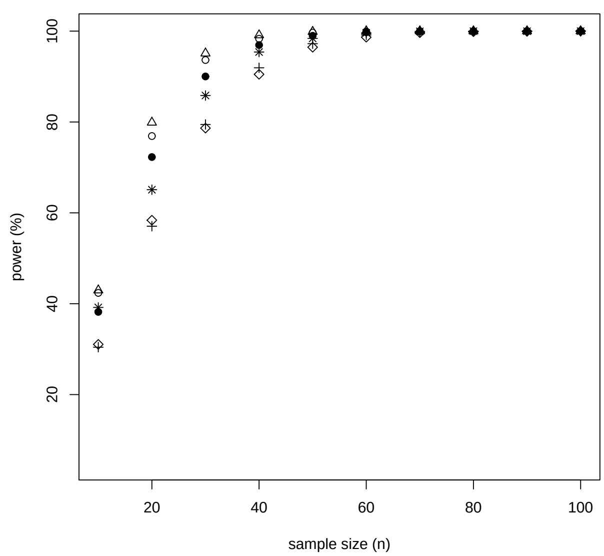
(a) Uniform



(b) Beta

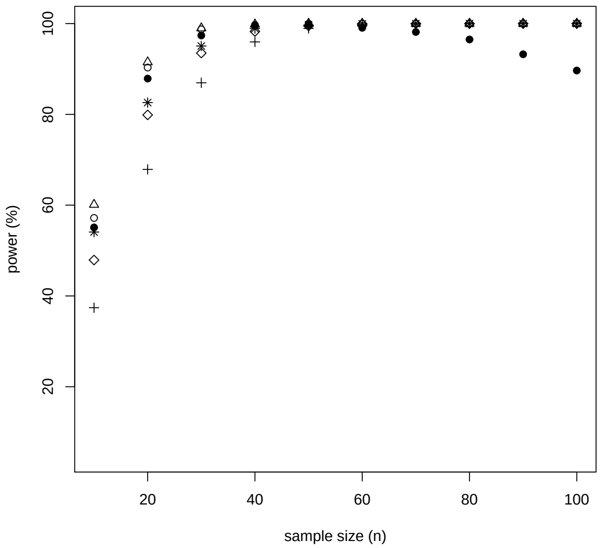


(c) Bi-modal

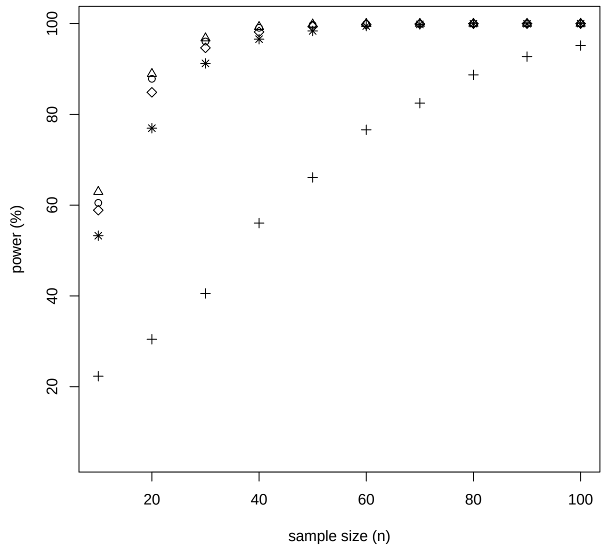


(d) Exponential

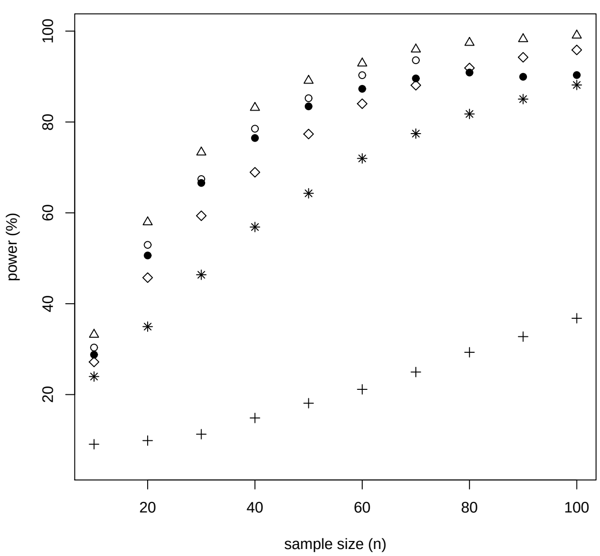
Figure 1: Approximate powers (%) of six tests, i.e. the traditional Kolmogorov-Smirnov (◇), modified Kolmogorov-Smirnov (+), Anderson-Darling (○), Cramer-Von Mises (●), Shapiro-Francia (△) and Pearson Chi-square (∗).



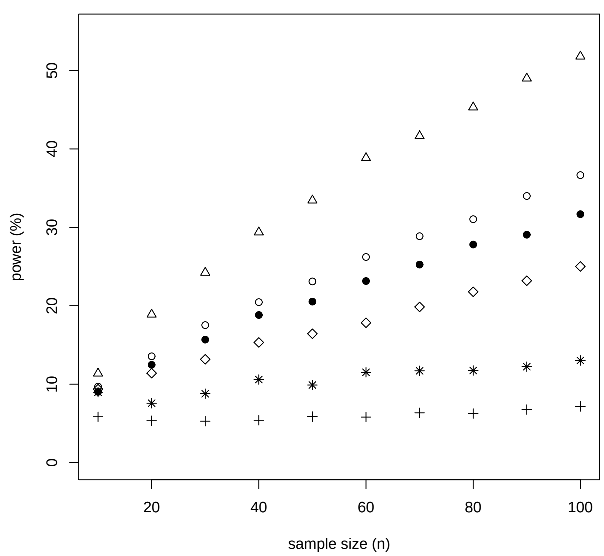
(a) Log-normal



(b) t_1

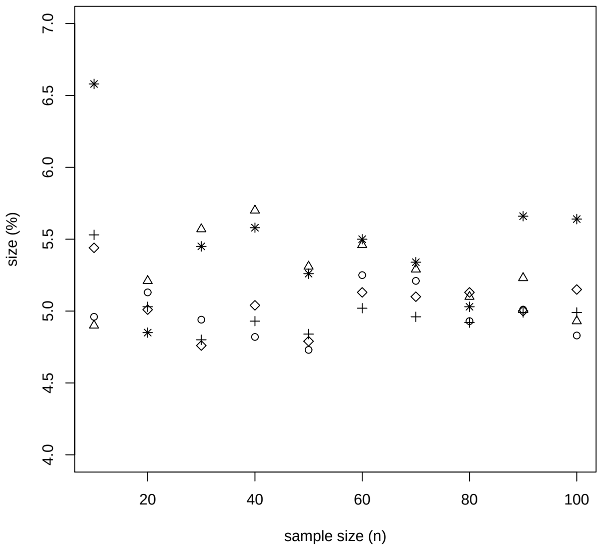


(c) t_2

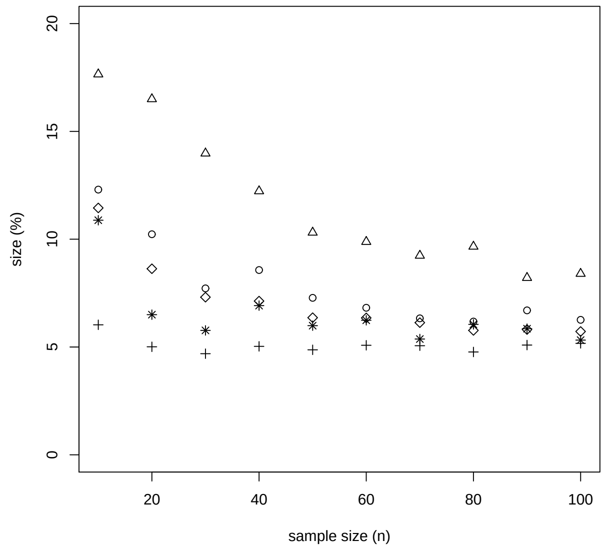


(d) t_6

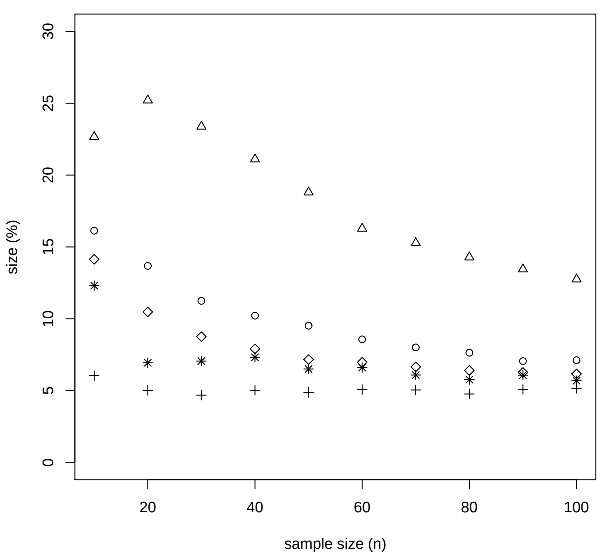
Figure 1: **CONTINUED** $\cdots$ approximate powers (%) of six tests, i.e. the traditional Kolmogorov-Smirnov ($\diamond$), modified Kolmogorov-Smirnov ($+$), Anderson-Darling ($\circ$), Cramer-Von Mises ($\bullet$), Shapiro-Francia ($\triangle$) and Pearson Chi-square ($*$).



(a) Normal



(b) Normal with C=3.0



(c) Normal with C=3.25

Figure 2: Rejection probabilities (%) of six tests, i.e. the traditional Kolmogorov-Smirnov (◇), modified Kolmogorov-Smirnov (+), Anderson-Darling (○), Cramer-Von Mises (●), Shapiro-Francia (△) and Pearson Chi-square (∗).

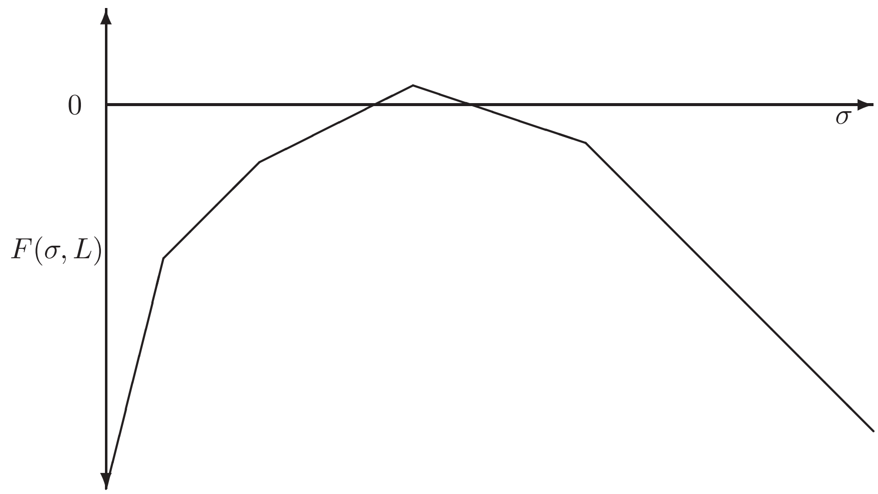


Figure 3: The Function $F(\sigma, L)$